

Random approximation and the vertex index of convex bodies

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Abstract

We prove that there exists an absolute constant $\alpha > 1$ with the following property: if K is a convex body in \mathbb{R}^n whose center of mass is at the origin, then a random subset $X \subset K$ of cardinality $\text{card}(X) = \lceil \alpha n \rceil$ satisfies with probability greater than $1 - e^{-n}$

$$K \subseteq c_1 n \text{conv}(X),$$

where $c_1 > 0$ is an absolute constant. As an application we show that the vertex index of any convex body K in \mathbb{R}^n is bounded by $c_2 n^2$, where $c_2 > 0$ is an absolute constant, thus extending an estimate of Bezdek and Litvak for the symmetric case.

1 Introduction

The starting point of this article is the following result of Barvinok from [3]: If $C \subset \mathbb{R}^n$ is a compact set then, for every $d > 1$ there exists a subset $X \subseteq C$ of cardinality $\text{card}(X) \leq dn$ such that for any $z \in \mathbb{R}^n$ we have

$$(1.1) \quad \max_{x \in X} |\langle z, x \rangle| \leq \max_{x \in C} |\langle z, x \rangle| \leq \gamma_d \sqrt{n} \max_{x \in X} |\langle z, x \rangle|,$$

where $\gamma_d := \frac{\sqrt{d+1}}{\sqrt{d-1}}$. For the proof of this fact, Barvinok assumes that the Euclidean unit ball B_2^n is the ellipsoid of minimal volume containing the convex hull $\text{conv}(C)$ of C , and makes essential use of a theorem of Batson, Spielman and Srivastava [4] on extracting an approximate John's decomposition with few vectors from a John's decomposition of the identity. From (1.1) one can easily conclude that if K is an origin symmetric convex body in \mathbb{R}^n then for any $d > 1$ there exist $N \leq dn$ points $x_1, \dots, x_N \in K$ such that

$$(1.2) \quad \text{absconv}(\{x_1, \dots, x_N\}) \subseteq K \subseteq \gamma_d \sqrt{n} \text{absconv}(\{x_1, \dots, x_N\}).$$

A generalization of Barvinok's lemma was recently obtained by the first named author in [7]: There exists an absolute constant $\alpha > 1$ with the following property: if K is a convex body whose minimal volume ellipsoid is the Euclidean unit ball, then there exist $N \leq \alpha n$ points $x_1, \dots, x_N \in K \cap S^{n-1}$ such that

$$(1.3) \quad K \subseteq B_2^n \subseteq cn^{3/2} \text{conv}(X),$$

where $c > 0$ is an absolute constant. The proof involves a more delicate theorem of Srivastava from [21]. Using (1.3) one can establish the following “quantitative diameter version” of Helly's theorem (see [7]): If $\{P_i : i \in I\}$ is a finite family of convex bodies in \mathbb{R}^n with $\text{diam}(\bigcap_{i \in I} P_i) = 1$, then there exist $s \leq \alpha n$ and $i_1, \dots, i_s \in I$ such that

$$(1.4) \quad \text{diam}(P_{i_1} \cap \dots \cap P_{i_s}) \leq cn^{3/2},$$

where $c > 0$ is an absolute constant. Our first main result provides a random version of (1.3) with an improved dependence on the dimension.

Theorem 1.1. *There exists an absolute constant $\alpha > 1$ with the following property: if K is a convex body in \mathbb{R}^n whose center of mass is at the origin, if $N = \lceil \alpha n \rceil$ and if x_1, \dots, x_N are independent random points uniformly distributed in K then, with probability greater than $1 - e^{-n}$ we have*

$$(1.5) \quad K \subseteq c_1 n \operatorname{conv}(\{x_1, \dots, x_N\}),$$

where $c_1 > 0$ is an absolute constant.

For the proof we may assume that K is an isotropic convex body (see Section 2 for background information) and we use the so-called one-sided L_q -centroid bodies of K ; these are the convex bodies $Z_q^+(K)$, $q \geq 1$, with support functions

$$(1.6) \quad h_{Z_q^+(K)}(y) = \left(2 \int_K \langle x, y \rangle_+^q dx \right)^{1/q},$$

where $a_+ = \max\{a, 0\}$. We show that if $N \geq \alpha n$, where $\alpha > 1$ is an absolute constant, then N independent random points x_1, \dots, x_N uniformly distributed in K satisfy

$$(1.7) \quad \operatorname{conv}(\{x_1, \dots, x_N\}) \supseteq c_1 Z_2^+(K) \supseteq c_2 L_K B_2^n$$

with probability greater than $1 - \exp(-n)$, where $c_1, c_2 > 0$ are absolute constants. Since K is contained in $(n+1)L_K B_2^n$, Theorem 1.1 follows.

We were led to Theorem 1.1 by the question to estimate the vertex index of a not necessarily symmetric n -dimensional convex body. The vertex index of a symmetric convex body K in \mathbb{R}^n was introduced in [5] as follows:

$$(1.8) \quad \operatorname{vi}(K) = \inf \left\{ \sum_{j=1}^N \|y_j\|_K : K \subseteq \operatorname{conv}(\{y_1, \dots, y_N\}) \right\},$$

where $\|\cdot\|_K$ is the norm with unit ball K in \mathbb{R}^n . This index is closely related to the illumination parameter of a convex body and to a well-known conjecture of Boltyanski and Hadwiger about covering of an n -dimensional convex body by 2^n smaller positively homothetic copies (see [5] and [11]). Bezdek and Litvak proved that

$$(1.9) \quad \frac{c_1 n^{3/2}}{\operatorname{ovr}(K)} \leq \operatorname{vi}(K) \leq c_2 n^{3/2},$$

where $c_1, c_2 > 0$ are absolute constants and $\operatorname{ovr}(K)$ is the outer volume ratio of K (see Section 2 for the definition). To the best of our knowledge the notion of vertex index has not been studied in the not necessarily symmetric case. A way to define it for an arbitrary convex body K in \mathbb{R}^n is to consider first any $z \in \operatorname{int}(K)$ and to set

$$(1.10) \quad \operatorname{vi}_z(K) = \inf \left\{ \sum_{j=1}^N p_{K,z}(y_j) : K \subseteq \operatorname{conv}(\{y_1, \dots, y_N\}) \right\},$$

where

$$(1.11) \quad p_{K,z}(x) = p_{K-z}(x) = \inf\{t > 0 : x \in t(K - z)\}$$

is the Minkowski functional of K with respect to z . Then, one may define the (generalized) vertex index of K by

$$(1.12) \quad \operatorname{vi}(K) = \operatorname{vi}_{\operatorname{bar}(K)}(K),$$

where $\operatorname{bar}(K)$ is the center of mass of K . With this definition, we clearly have $\operatorname{vi}(K) = \operatorname{vi}(K - \operatorname{bar}(K))$, and hence we may restrict our attention to centered convex bodies (i.e. convex bodies whose center of mass is at the origin). In Section 4 we establish some elementary properties of this index and using Theorem 1.1 we obtain the following general estimate.

Theorem 1.2. *There exist two absolute constants $c_1, c_2 > 0$ such that for every $n \geq 2$ and for every centered convex body K in \mathbb{R}^n ,*

$$(1.13) \quad \frac{c_1 n^{3/2}}{\text{ovr}(\text{conv}(K, -K))} \leq \text{vi}(K) \leq c_2 n^2.$$

A natural question, which is closely related to Theorem 1.1, is to fix $N \geq \alpha n$ and to ask for the largest value $t(N, n)$ for which N independent random points x_1, \dots, x_N uniformly distributed in K satisfy

$$(1.14) \quad \text{conv}(\{x_1, \dots, x_N\}) \supseteq t(N, n) K$$

with probability “exponentially close” to 1. A sharp answer to this question would unify Theorem 1.1 and the following result from [10] which deals with the case where N is exponential in n : For every $\delta \in (0, 1)$ there exists $n_0 = n_0(\delta)$ such that if $n \geq n_0$, if $C \log n / n \leq \gamma \leq 1$ and if K is a centered convex body in \mathbb{R}^n , then $N = \exp(\gamma n)$ independent random points x_1, \dots, x_N chosen uniformly from K satisfy with probability greater than $1 - \delta$

$$(1.15) \quad K \supseteq \text{conv}(\{x_1, \dots, x_N\}) \supseteq c(\delta) \gamma K,$$

where $c(\delta)$ is a constant depending on δ . We prove the following.

Theorem 1.3. *Let $\beta \in (0, 1)$. There exist a constant $\alpha = \alpha(\beta) > 1$ depending only on β and an absolute constant $c_1 > 0$ with the following property: if K is a centered convex body in \mathbb{R}^n , if $\alpha n \leq N \leq e^n$ and if x_1, \dots, x_N are independent random points uniformly distributed in K , then*

$$(1.16) \quad \text{conv}(\{x_1, \dots, x_N\}) \supseteq \frac{c_1 \beta \log(N/n)}{n} K.$$

with probability greater than $1 - e^{-N^{1-\beta} n^\beta}$.

In fact, Theorem 1.1 is a special case of Theorem 1.3. The proof of both theorems is given in Section 3.

2 Notation and background

We work in \mathbb{R}^n , which is equipped with a Euclidean structure $\langle \cdot, \cdot \rangle$. We denote by $\|\cdot\|_2$ the corresponding Euclidean norm, and write B_2^n for the Euclidean unit ball and S^{n-1} for the unit sphere. Volume is denoted by $|\cdot|$. We use the same notation $|X|$ for the cardinality of a finite set X . We write ω_n for the volume of B_2^n and σ for the rotationally invariant probability measure on S^{n-1} .

The letters c, c', c_1, c_2, \dots denote absolute positive constants which may change from line to line. Whenever we write $a \leq b$, we mean that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \leq b \leq c_2 a$. Also, if $K, L \subseteq \mathbb{R}^n$ we will write $K \simeq L$ if there exist absolute constants $c_1, c_2 > 0$ such that $c_1 K \subseteq L \subseteq c_2 K$.

We refer to the book of Schneider [20] for basic facts from the Brunn-Minkowski theory and to the book of Artstein-Avidan, Giannopoulos and V. Milman [1] for basic facts from asymptotic convex geometry.

A convex body in \mathbb{R}^n is a compact convex subset K of \mathbb{R}^n with non-empty interior. We say that K is symmetric if $x \in K$ implies that $-x \in K$, and that K is centered if its center of mass

$$(2.1) \quad \text{bar}(K) = \frac{1}{|K|} \int_K x \, dx$$

is at the origin. The circumradius of K is the radius of the smallest ball which is centered at the origin and contains K :

$$(2.2) \quad R(K) = \max\{\|x\|_2 : x \in K\}.$$

If $0 \in \text{int}(K)$ then the polar body K° of K is defined by

$$(2.3) \quad K^\circ := \{y \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } x \in K\},$$

and the Minkowski functional of K is defined by

$$(2.4) \quad p_K(x) = \inf\{t > 0 : x \in tK\}.$$

Recall that p_K is subadditive and positively homogeneous.

We say that a convex body K is in John's position if the ellipsoid of maximal volume inscribed in K is the Euclidean unit ball B_2^n . John's theorem (see [1, Chapter 2]) states that K is in John's position if and only if $B_2^n \subseteq K$ and there exist $v_1, \dots, v_m \in \text{bd}(K) \cap S^{n-1}$ (contact points of K and B_2^n) and positive real numbers a_1, \dots, a_m such that

$$(2.5) \quad \sum_{j=1}^m a_j v_j = 0$$

and the identity operator I_n is decomposed in the form

$$(2.6) \quad I_n = \sum_{j=1}^m a_j v_j \otimes v_j,$$

where $(v_j \otimes v_j)(y) = \langle v_j, y \rangle v_j$. We say that a convex body K is in Löwner's position if the ellipsoid of minimal volume containing K is the Euclidean unit ball B_2^n . One can check that this holds true if and only if K° is in John's position; in particular, we have a decomposition of the identity similar to (2.6). The outer volume ratio of a convex body K in \mathbb{R}^n is the quantity

$$(2.7) \quad \text{ovr}(K) = \inf \left\{ \left(\frac{|\mathcal{E}|}{|K|} \right)^{1/n} : \mathcal{E} \text{ is an ellipsoid and } K \subseteq \mathcal{E} \right\}.$$

If K is in Löwner's position then $(|B_2^n|/|K|)^{1/n} = \text{ovr}(K)$.

A convex body K in \mathbb{R}^n is called isotropic if it has volume 1, it is centered, and its inertia matrix is a multiple of the identity matrix: there exists a constant $L_K > 0$ such that

$$(2.8) \quad \int_K \langle x, \theta \rangle^2 dx = L_K^2$$

for every θ in the Euclidean unit sphere S^{n-1} . It is known that if K is isotropic then

$$(2.9) \quad cL_K B_2^n \subseteq K \subseteq (n+1)L_K B_2^n,$$

where $c > 0$ is an absolute constant. The hyperplane conjecture asks if there exists an absolute constant $C > 0$ such that

$$(2.10) \quad L_n := \max\{L_K : K \text{ is isotropic in } \mathbb{R}^n\} \leq C$$

for all $n \geq 1$. Bourgain proved in [6] that $L_n \leq c\sqrt[4]{n} \log n$, while Klartag [14] obtained the bound $L_n \leq c\sqrt[4]{n}$. A second proof of Klartag's bound appears in [15]. We refer the reader to the article of V. Milman and Pajor [17] and to the book [8] for an updated exposition of isotropic log-concave measures and more information on the hyperplane conjecture.

The L_q -centroid body $Z_q(K)$ of K is the centrally symmetric convex body with support function

$$(2.11) \quad h_{Z_q(K)}(y) = \left(\int_K |\langle x, y \rangle|^q dx \right)^{1/q}.$$

Note that K is isotropic if and only if it is centered and $Z_2(K) = L_K B_2^n$. Also, if $T \in SL(n)$ then $Z_q(T(K)) = T(Z_q(K))$. From Hölder's inequality it follows that $Z_1(K) \subseteq Z_p(K) \subseteq Z_q(K) \subseteq Z_\infty(K)$ for all $1 \leq p \leq q \leq \infty$, where $Z_\infty(K) = \text{conv}(K, -K)$. Using Borell's lemma (see [8, Chapter 1]) one can check that

$$(2.12) \quad Z_q(K) \subseteq \bar{c}_1 \frac{q}{p} Z_p(K)$$

for all $1 \leq p < q$, where $\bar{c}_1 > 0$ is an absolute constant. In particular, if K is isotropic then

$$(2.13) \quad R(Z_q(K)) \leq \bar{c}_1 q L_K.$$

One can also check that if K is centered, then $Z_q(K) \supseteq c_2 Z_\infty(K)$ for all $q \geq n$. For a proof of all these assertions see [8, Chapter 5]. The class of L_q -centroid bodies of K was introduced (with a different normalization) by Lutwak, Yang and Zhang in [16]. An asymptotic approach to this family was developed by Paouris in [18] and [19].

For the proof of Theorem 1.3 we generalize the arguments from [9] who used L_q -centroid bodies in order to describe the asymptotic shape of the absolute convex hull of N random points chosen from a convex body. The use of one-sided L_q -centroid bodies allows one to consider the convex hull itself.

3 Random approximation of convex bodies

Let K be a centered convex body of volume 1 in \mathbb{R}^n . For every $q \geq 1$ we consider the one-sided L_q -centroid body $Z_q^+(K)$ of K with support function

$$(3.1) \quad h_{Z_q^+(K)}(y) = \left(2 \int_K \langle x, y \rangle_+^q dx \right)^{1/q},$$

where $a_+ = \max\{a, 0\}$. When K is symmetric, it is clear that $Z_q^+(K) = Z_q(K)$. In any case, we easily verify that

$$(3.2) \quad Z_q^+(K) \subseteq 2^{1/q} Z_q(K).$$

Note that $Z_q^+(K) \subseteq 2^{1/q} K$ for all $q \geq 1$. Using Grünbaum's lemma (see [1, Proposition 1.5.16]) one can check that if $1 \leq q \leq r < \infty$ then

$$(3.3) \quad \left(\frac{2}{e} \right)^{\frac{1}{q} - \frac{1}{r}} Z_q^+(K) \subseteq Z_r^+(K) \subseteq \frac{Cr}{q} \left(\frac{2e-2}{e} \right)^{\frac{1}{q} - \frac{1}{r}} Z_q^+(K),$$

where $C > 0$ is an absolute constant. The next lemma is due to Guédon and E. Milman (see [13]).

Lemma 3.1. *There exists an absolute constant $\bar{c}_0 > 0$ such that, for every isotropic convex body K in \mathbb{R}^n ,*

$$(3.4) \quad Z_2^+(K) \supseteq \bar{c}_0 L_K B_2^n.$$

Equivalently, for any $\theta \in S^{n-1}$,

$$(3.5) \quad h_{Z_2^+(K)}(\theta) = \left(2 \int_K \langle x, \theta \rangle_+^2 dx \right)^{1/2} \geq \bar{c}_0 L_K.$$

We also need the next lemma, which appears in [13] (see also [8, Theorem 13.2.7]).

Lemma 3.2. *Let K be a centered convex body of volume 1 in \mathbb{R}^n . We fix $\theta \in S^{n-1}$ and define $f_\theta(t) = |K \cap \{x : \langle x, \theta \rangle = t\}|$, $t \in \mathbb{R}$. Then,*

$$(3.6) \quad \left(\frac{2}{e^2} \right)^{1/q} \left(\frac{\Gamma(n)\Gamma(q+1)}{\Gamma(n+q+1)} \right)^{1/q} h_K(\theta) \leq h_{Z_q^+(K)}(\theta) \leq 2^{1/q} h_K(\theta).$$

Proof. We sketch the proof of the left hand side inequality. Let

$$(3.7) \quad H_\theta^+ = \{x \in \mathbb{R}^n : \langle x, \theta \rangle \geq 0\}.$$

First observe that, by the Brunn-Minkowski inequality, $f_\theta^{\frac{1}{n-1}}$ is concave on its support, and hence we have

$$(3.8) \quad f_\theta(t) \geq \left(1 - \frac{t}{h_K(\theta)}\right)^{n-1} f_\theta(0)$$

for all $t \in [0, h_K(\theta)]$. Therefore,

$$(3.9) \quad \begin{aligned} h_{Z_q^+(K)}^q(\theta) &= 2 \int_0^{h_K(\theta)} t^q f_\theta(t) dt \geq 2 \int_0^{h_K(\theta)} t^q \left(1 - \frac{t}{h_K(\theta)}\right)^{n-1} f_\theta(0) dt \\ &= 2f_\theta(0) h_K^{q+1}(\theta) \int_0^1 s^q (1-s)^{n-1} ds \\ &= \frac{\Gamma(n)\Gamma(q+1)}{\Gamma(q+n+1)} 2f_\theta(0) h_K^{q+1}(\theta). \end{aligned}$$

Observe that

$$(3.10) \quad 2f_\theta(0)h_K(\theta) = \frac{f_\theta(0)}{\|f_\theta\|_\infty} 2\|f_\theta\|_\infty h_K(\theta) \geq \frac{f_\theta(0)}{\|f_\theta\|_\infty} (2|K \cap H_\theta^+|).$$

We know that $\|f_\theta\|_\infty \leq ef_\theta(0)$ by a result of Fradelizi (see e.g. [8, Theorem 2.2.2]) and that $|K \cap H_\theta^+| \geq e^{-1}$ by Grünbaum's lemma (see [1, Proposition 1.5.16]). Combining the above we get the result. \square

Theorem 1.3 and Theorem 1.1 will follow from the next fact, which generalizes work of Dafnis, Giannopoulos and Tsolomitis [9] to the not necessarily symmetric setting.

Theorem 3.3. *Let $\beta \in (0, 1)$. There exist a constant $\alpha = \alpha(\beta) > 1$ depending only on β and absolute constants $c_1, c_2 > 0$ with the following property: if K is a centered convex body in \mathbb{R}^n , if $N \geq \alpha n$ and if x_1, \dots, x_N are independent random points uniformly distributed in K then there exists $q \geq c_1 \beta \log(N/n)$ such that*

$$(3.11) \quad \text{conv}(\{x_1, \dots, x_N\}) \supseteq c_2 Z_q^+(K)$$

with probability greater than $1 - e^{-N^{1-\beta} n^\beta}$.

Our proof of (3.11) is using the family of one-sided L_q -centroid bodies of K . In particular, we need the following estimate.

Lemma 3.4. *There exists an absolute constant $C > 1$ with the following property: for every $n \geq 1$, for every centered convex body K in \mathbb{R}^n and for every $q \geq 2$,*

$$(3.12) \quad \inf_{\theta \in S^{n-1}} \mu_K \left(\left\{ x : \langle x, \theta \rangle > \frac{1}{2} h_{Z_q^+(K)}(\theta) \right\} \right) \geq C^{-q}.$$

Proof. Let K be a centered convex body in \mathbb{R}^n , let $q \geq 2$ and let $\theta \in S^{n-1}$. We apply the Paley-Zygmund inequality

$$(3.13) \quad \mathbb{P}(g \geq t\mathbb{E}(g)) \geq (1-t)^2 \frac{[\mathbb{E}(g)]^2}{\mathbb{E}(g^2)}$$

for the non-negative random variable

$$(3.14) \quad g_\theta(x) = 2\langle x, \theta \rangle_+^q$$

on (K, μ_K) , where μ_K is Lebesgue measure on K . Applying (3.3) with $r = 2q$ we see that

$$(3.15) \quad \mathbb{E}(g_\theta^2) = h_{Z_{2q}^+(K)}^{2q}(\theta) \leq C_1^q h_{Z_q^+(K)}^{2q}(\theta) = C_1^q [\mathbb{E}(g_\theta)]^2,$$

where $C_1 > 0$ is an absolute constant. From (3.13) we get

$$(3.16) \quad \begin{aligned} \mu_K(\{x : \langle x, \theta \rangle > t h_{Z_q^+(K)}(\theta)\}) &= \mu_K(\{x : \langle x, \theta \rangle > t [\mathbb{E}(g_\theta)]^{1/q}\}) = \mu_K(\{x : \langle x, \theta \rangle_+ > t [\mathbb{E}(g_\theta)]^{1/q}\}) \\ &= \mu_K(\{x : \langle x, \theta \rangle_+^q > t^q \mathbb{E}(g_\theta)\}) = \mu_K(\{x : g_\theta(x) > 2t^q \mathbb{E}(g_\theta)\}) \\ &\geq (1 - 2t^q)^2 \frac{[\mathbb{E}(g_\theta)]^2}{\mathbb{E}(g_\theta^2)} \geq \frac{(1 - 2t^q)^2}{C_1^q} \end{aligned}$$

for every $t \in (0, 2^{-\frac{1}{q}})$. Choosing $t = \frac{1}{2}$ we get the lemma with $C = 4C_1$. \square

Proof of Theorem 3.3. Let $q \geq 2$ and consider the random polytope $C_N := \text{conv}\{x_1, \dots, x_N\}$. With probability equal to one, C_N has non-empty interior and, for every $J = \{j_1, \dots, j_n\} \subset \{1, \dots, N\}$, the points x_{j_1}, \dots, x_{j_n} are affinely independent. Write H_J for the affine subspace determined by x_{j_1}, \dots, x_{j_n} and H_J^+ , H_J^- for the two closed halfspaces whose bounding hyperplane is H_J .

If $\frac{1}{2}Z_q^+(K) \not\subseteq C_N$, then there exists $x \in \frac{1}{2}Z_q^+(K) \setminus C_N$, and hence, there is a facet of C_N defining some affine subspace H_J as above that satisfies the following: either $x \in H_J^-$ and $C_N \subset H_J^+$, or $x \in H_J^+$ and $C_N \subset H_J^-$. Observe that, for every J , the probability of each of these two events is bounded by

$$(3.17) \quad \left(\sup_{\theta \in S^{n-1}} \mu_K\left(\left\{x : \langle x, \theta \rangle \leq \frac{1}{2}h_{Z_q^+(K)}(\theta)\right\}\right) \right)^{N-n} \leq (1 - C^{-q})^{N-n},$$

where $C > 0$ is the constant in Lemma 3.4. It follows that

$$(3.18) \quad \mathbb{P}\left(\frac{1}{2}Z_q^+(K) \not\subseteq C_N\right) \leq 2 \binom{N}{n} (1 - C^{-q})^{N-n}.$$

Since $\binom{N}{n} \leq \left(\frac{eN}{n}\right)^n$, this probability is smaller than $\exp(-N^{1-\beta}n^\beta)$ if

$$(3.19) \quad \left(\frac{2eN}{n}\right)^n (1 - C^{-q})^{N-n} < \left(\frac{2eN}{n}\right)^n e^{-C^{-q}(N-n)} < \exp(-N^{1-\beta}n^\beta),$$

and the second inequality is satisfied if

$$(3.20) \quad \frac{N}{n} - 1 > C^q \left[\left(\frac{N}{n}\right)^{1-\beta} + \log\left(\frac{2eN}{n}\right) \right].$$

We choose $q = \frac{\beta}{2 \log C} \log\left(\frac{N}{n}\right)$ and $\alpha_1(\beta) := C^{4/\beta}$. Note that if $N \geq \alpha_1(\beta)n$ then $q \geq 2$ if and that (3.20) becomes

$$(3.21) \quad \frac{N}{n} - 1 > \left(\frac{N}{n}\right)^{1-\frac{\beta}{2}} + \left(\frac{N}{n}\right)^{\frac{\beta}{2}} \log\left(\frac{2eN}{n}\right).$$

Since

$$(3.22) \quad \lim_{t \rightarrow +\infty} \left[t - 1 - t^{1-\frac{\beta}{2}} - t^{\frac{\beta}{2}} \log(2et) \right] = +\infty,$$

we may find $\alpha_2(\beta)$ such that (3.21) is satisfied for all $N \geq \alpha_2(\beta)n$. Setting $\alpha = \max\{\alpha_1(\beta), \alpha_2(\beta)\}$ we see that the assertion of the theorem is satisfied with probability greater than $1 - e^{-N^{1-\beta}n^\beta}$ for all $N \geq \alpha n$, with $q \geq c_2 \beta \log\left(\frac{N}{n}\right)$, where $c_2 > 0$ is an absolute constant. \square

Proof of Theorem 1.3. Let $\beta \in (0, 1)$ and let $\alpha = \alpha(\beta)$ be the constant from Theorem 3.3. Let $\alpha n \leq N \leq e^n$ and let x_1, \dots, x_N be independent random points uniformly distributed in K . Applying Lemma 3.2 with $q = n$ we see that $h_{Z_n^+(K)} \geq c_1 h_K(\theta)$ for all $\theta \in S^{n-1}$, and hence

$$(3.23) \quad Z_n^+(K) \supseteq c_1 K,$$

where $c_1 > 0$ is an absolute constant. From Theorem 3.3 we know that if $q = c_2 \beta \log(N/n)$ (note also that $q \leq n$) then

$$(3.24) \quad C_N = \text{conv}(\{x_1, \dots, x_N\}) \supseteq c_3 Z_q^+(K)$$

with probability greater than $1 - \exp(-N^{1-\beta} n^\beta)$, where $c_2, c_3 > 0$ are absolute constants. From (3.3) we see that

$$(3.25) \quad Z_n^+(K) \subseteq \frac{c_4 n}{q} \left(\frac{2e-2}{e} \right)^{\frac{1}{q} - \frac{1}{n}} Z_q^+(K) \subseteq \frac{2c_4 n}{q} Z_q^+(K),$$

where $c_4 > 0$ is an absolute constant. Combining the above we get that

$$(3.26) \quad C_N = \text{conv}(\{x_1, \dots, x_N\}) \supseteq \frac{c_5 q}{n} K \supseteq \frac{c_6 \beta \log(N/n)}{n} K$$

with probability greater than $1 - \exp(-N^{1-\beta} n^\beta)$, where $c_5, c_6 > 0$ are absolute constants. \square

Choosing $N = \lceil \alpha n \rceil$ in Theorem 1.3 we immediately get Theorem 1.1:

Theorem 3.5. *There exists an absolute constant $\alpha > 1$ with the following property: if K is a centered convex body in \mathbb{R}^n then a random subset $X \subset K$ of cardinality $\text{card}(X) = \lceil \alpha n \rceil$ satisfies*

$$(3.27) \quad K \subseteq Cn \text{conv}(X)$$

with probability greater than $1 - e^{-n}$, where $C > 0$ is an absolute constant.

4 Generalized vertex index

Let K be a convex body in \mathbb{R}^n . From the definition of the vertex index that we gave in the introduction, we may clearly assume that K is centered, and then

$$(4.1) \quad \text{vi}(K) = \inf \left\{ \sum_{j=1}^N p_K(y_j) : K \subseteq \text{conv}(\{y_1, \dots, y_N\}) \right\},$$

where p_K is the Minkowski functional of K . Since every origin symmetric convex body is centered, our definition coincides with the one given by Bezdek and Litvak in [5] for the symmetric case.

It is also easy to check that the vertex index is invariant under invertible linear transformations. For every convex body K in \mathbb{R}^n and any $T \in GL(n)$ one has

$$(4.2) \quad \text{vi}(T(K)) = \text{vi}(K).$$

To see this, note that $T(K) \subseteq \text{conv}(\{y_1, \dots, y_N\})$ if and only if $K \subseteq \text{conv}(\{x_1, \dots, x_N\})$ where $T(x_j) = y_j$, therefore

$$\begin{aligned} (4.3) \quad \text{vi}(T(K)) &= \inf \left\{ \sum_{j=1}^N p_{T(K)}(T(x_j)) : K \subseteq \text{conv}(\{x_1, \dots, x_N\}) \right\} \\ &= \inf \left\{ \sum_{j=1}^N p_K(x_j) : K \subseteq \text{conv}(\{x_1, \dots, x_N\}) \right\} \\ &= \text{vi}(K). \end{aligned}$$

Another useful observation is that the vertex index is stable under a variant of the Banach-Mazur distance. Recall that the Banach-Mazur distance between two convex bodies K and L in \mathbb{R}^n is the quantity

$$(4.4) \quad d(K, L) = \inf\{t > 0 : T(L + y) \subseteq K + x \subseteq t(T(L + y))\},$$

where the infimum is over all $T \in GL(n)$ and $x, y \in \mathbb{R}^n$. Given two centered convex bodies K and L , we set

$$(4.5) \quad \tilde{d}(K, L) = \inf\{t > 0 : T(L) \subseteq K \subseteq tT(L)\},$$

where the infimum is over all $T \in GL(n)$. Note that if K and L are symmetric convex bodies then $\tilde{d}(K, L) = d(K, L)$. With this definition we easily check that if K and L are centered convex bodies in \mathbb{R}^n then

$$(4.6) \quad \text{vi}(K) \leq \tilde{d}(K, L) \text{vi}(L).$$

The main result of this section is the upper bound in Theorem 1.2.

Proposition 4.1. *There exists an absolute constant $C_1 > 0$ such that, for every $n \geq 2$ and for every centered convex body K in \mathbb{R}^n ,*

$$(4.7) \quad \text{vi}(K) \leq C_1 n^2.$$

Proof. We may assume that K is isotropic. By Theorem 3.5 we can find $N \leq \alpha n$ and $x_1, \dots, x_N \in K$ such that

$$(4.8) \quad K \subseteq Cn \text{conv}(\{x_1, \dots, x_N\}),$$

where $\alpha, C > 0$ are absolute constants. We set $y_j = Cn x_j$, $1 \leq j \leq N$. Then, $K \subseteq \text{conv}(\{y_1, \dots, y_N\})$ and $p_K(y_j) = Cn p_K(x_j) \leq Cn$, therefore

$$(4.9) \quad \text{vi}(K) \leq \sum_{j=1}^N p_K(y_j) \leq CnN \leq C\alpha n^2.$$

The result follows with $C_1 = C\alpha$. □

For the lower bound we just check that the argument of [5] remains valid in the not necessarily symmetric case.

Proposition 4.2. *There exists an absolute constant $c > 0$ such that, for every $n \geq 2$ and for every centered convex body K in \mathbb{R}^n ,*

$$(4.10) \quad \text{vi}(K) \geq \frac{cn^{3/2}}{\text{ovr}(\text{conv}(K, -K))}.$$

Proof. By the linear invariance of the vertex index we may assume that B_2^n is the ellipsoid of minimal volume which contains $\text{conv}(K, -K)$. In other words, $K \subseteq \text{conv}(K, -K) \subseteq B_2^n$ and

$$(4.11) \quad \left(\frac{|B_2^n|}{|\text{conv}(K, -K)|} \right)^{1/n} = \text{ovr}(\text{conv}(K, -K)).$$

For any $N \in \mathbb{N}$ and y_1, \dots, y_N such that $K \subseteq \text{conv}(\{y_1, \dots, y_N\})$ we consider the absolute convex hull $Q = \text{conv}(\{\pm y_1, \dots, \pm y_N\}) \supseteq \text{conv}(K, -K)$ of y_1, \dots, y_N . Then,

$$(4.12) \quad Q^\circ = \{x \in \mathbb{R}^n : |\langle x, y_j \rangle| \leq 1 \text{ for all } j = 1, \dots, N\},$$

and a result of Ball and Pajor [2] provides the lower bound

$$(4.13) \quad |Q^\circ| \geq \left(\frac{n}{\sum_{j=1}^N \|y_j\|_2} \right)^{1/n}$$

for its volume. Using the Blaschke-Santaló inequality we get

$$(4.14) \quad |\text{conv}(K, -K)| \leq |Q| \leq \frac{|B_2^n|^2}{|Q^\circ|} \leq |B_2^n|^2 \left(\frac{\sum_{j=1}^N \|y_j\|_2}{n} \right)^n.$$

It follows that

$$(4.15) \quad 1 \leq \left(\frac{|B_2^n|}{|\text{conv}(K, -K)|} \right)^{1/n} |B_2^n|^{1/n} \frac{\sum_{j=1}^N \|y_j\|_2}{n} \leq \frac{\text{ovr}(\text{conv}(K, -K))}{cn^{3/2}} \sum_{j=1}^N \|y_j\|_2$$

for some absolute constant $c > 0$. Since $K \subseteq B_2^n$, we have $\|y_j\|_2 \leq p_K(y_j)$ for all $j = 1, \dots, N$. Therefore,

$$(4.16) \quad \sum_{j=1}^N p_K(y_j) \geq \frac{cn^{3/2}}{\text{ovr}(\text{conv}(K, -K))},$$

and taking the infimum over all N and all such N -tuples (y_1, \dots, y_N) we get the lower bound for $\text{vi}(K)$. \square

Remark 4.3. The lower bound of Proposition 4.2 is not sharp, even in the symmetric case. Gluskin and Litvak [12] have proved that for every $n \geq 1$ there exists a symmetric convex body K in \mathbb{R}^n such that

$$(4.17) \quad \text{ovr}(K) \geq c \sqrt{\frac{n}{\log(2n)}} \quad \text{and} \quad \text{vi}(K) \geq cn^{3/2}.$$

It would be interesting to provide alternative lower bounds for $\text{vi}(K)$; in particular, it would be interesting to decide whether, in the non-symmetric case, the upper bound $\text{vi}(K) \leq Cn^2$ of Proposition 4.1 is sharp or not.

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